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Javier Cabello Sánchez
Departamento de Matemáticas de la Universidad de Extremadura
Instituto de investigación de Matemáticas de la Universidad de Extremadura

Extension of isometries, the geometry of unit spheres and a generalization of the curvature

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Abstract

The so-called Tingley's Problem, see [**5**] can be stated as follows:

QUESTION. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces and* $\tau : S_X \to S_Y$ *an onto isometry. Is* τ *the restriction of some linear isometry* $\widetilde{\tau} : X \to Y$*?*

This question is closely related to the Mazur-Ulam Theorem, ([**3**]), in the sense that every onto isometry $\widetilde{\tau} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is linear provided $\widetilde{\tau}(0) = 0$ and to Mankiewicz Theorem ([**2**]), that ensures that every onto isometry $\tau : F_X \to F_Y$ between connected open subsets, respectively, of $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ extends to an affine isometry $\widetilde{\tau} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$. Namely, the relation with Mazur-Ulam Theorem has led to state that some normed space X has the Mazur-Ulam Property if every $\tau : S_X \to S_Y$ extends linearly, no matter which space is Y, see, e.g., [**1**]. In this talk we will expose some ideas that may solve the problem with tools closer to the Geometry realm than to the Mathematical Analysis one. These can be found in [**4**]. Our current efforts are mainly focused on the quest of metric invariants in the spheres

of normed spaces, with special interest in two-dimensional spaces where we have managed to translate the problem to the following:

QUESTION. *Let* $(\mathbb{R}^2, \|\cdot\|_X)$ *and* $(\mathbb{R}^2, \|\cdot\|_Y)$ *be two-dimensional normed spaces such that* $(1,0),(0,1) \in S_X \cap S_Y$ *and* $\tau : S_X \to S_Y$ *an onto isometry with* $\tau(1,0) = (1,0), \tau(0,1) = (0,1)$*. Does* S_Y *equal* S_X*?*

This way to state the problem has led to some advances we will expose. On the other hand, we have found a surprising metric invariant that can be seen as a generalization of the concept of curvature of a planar curve and that may lead to intriguing questions. Namely,

DEFINITION. Let $(X, \|\cdot\|_X)$ be a normed space, $\gamma : [0,1] \to X$ a curve and $x = \gamma(t) \in X$ for some $0 < t < 1$. Suppose that there is some $c > 0$ such that $\gamma \cap (x + \delta S_X)$ contains exactly two points for every $0 < \delta < c$. We define the curvature of γ at x measured with $\|\cdot\|_X$ as the following limit, whenever it exists:

$$\mathcal{K}^{\gamma}_{\|\cdot\|_X}(x) = \sqrt{\lim_{\delta \to 0} \frac{\delta - \|a - a'\|_X/2}{(\delta/2)^3}} = 2\sqrt{\lim_{a,a' \to x} \frac{2\|x-a\|_X - \|a-a'\|_X}{(\|x-a\|_X)^3}}, \tag{1}$$

where $a \neq a'$ are the only points in γ such that $\|x - a\|_X = \|x - a'\|_X = \delta$.

We have proved the following:

THEOREM. *The curvature of every sphere* $(\alpha, \beta) + \lambda S_2 \subset \mathbb{R}^2$*, measured with* $\|\cdot\|_2$ *(in the sense of* (1)*) agrees in every point with* $\frac{1}{\lambda}$*.*

References

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